

# HARNACK TYPE INEQUALITY ON RIEMANNIAN MANIFOLDS OF DIMENSIONS 4, 5 AND 6.

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**ABSTRACT.** We give some estimates of type  $\sup \times \inf$  on Riemannian manifold of dimensions 4, 5 and 6 for the Yamabe type equation.

## 1. INTRODUCTION AND MAIN RESULTS

In this paper, we deal with the following Yamabe type equation in dimensions  $n = 4, 5, 6$

$$-\Delta_g u + \bar{h}(x)u = n(n-2)u^{N-1}, \quad u > 0, \quad \text{and} \quad N = \frac{n+2}{n-2}. \quad (1)$$

where  $\bar{h}$  is a continuous function. In the case  $\frac{4(n-1)\bar{h}}{n-2} = R_g$  the scalar curvature, (1) is the Yamabe equation. Here, we assume  $\bar{h}$  a bounded function and  $h_0 = \|\bar{h}\|_{L^\infty(M)}$ . The equation (1) has been well studied when  $M = \Omega \subset \mathbb{R}^n$  open, or  $M = \mathbb{S}_n$ , see for example, [2]-[4], [12], [16] and references therein, where sup-inf inequality or Harnack type inequalities are derived.

The corresponding equation in two dimensions on open set  $\Omega$  of  $\mathbb{R}^2$ , is

$$-\Delta u = V(x)e^u, \quad (2)$$

The equation (2) has also been studied by many authors and we can find important results about a priori estimates in [9], [10], [13], [17], and [21]. In particular, in [10] we have the interior estimate

$$\sup_K u \leq c = c(\inf_\Omega V, \|V\|_{L^\infty(\Omega)}, \inf_\Omega u, K, \Omega).$$

And, precisely, in [9], [13], [17], and [21], we have

$$C \sup_K u + \inf_\Omega u \leq c = c(\inf_\Omega V, \|V\|_{L^\infty(\Omega)}, K, \Omega),$$

and,

$$\sup_K u + \inf_\Omega u \leq c = c(\inf_\Omega V, \|V\|_{C^\alpha(\Omega)}, K, \Omega).$$

where  $K$  is a compact subset of  $\Omega$ ,  $C$  is a positive constant which depends on  $\frac{\inf_\Omega V}{\sup_\Omega V}$ , and,  $\alpha \in (0, 1]$ .

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When  $M$  is a compact Riemannian manifold, some compactness results have been proved for the equation (1), see [19]. In [19], Li and Zhu proved that the energy is bounded and if we assume that  $M$  not diffeomorphic to the three sphere, then the solutions are uniformly bounded. To prove this result, they use the positive mass theorem. For general Riemannian manifold  $M$  of dimensions 3 and 4, not necessarily compact Li and Zhang [18] proved that the product  $\sup \times \inf$  is bounded. The first author obtained Harnack type inequality for the solutions of

$$-8\Delta u + R_g u = V(x)u^5, \quad u > 0 \quad (3)$$

under certain conditions on  $V$  in [5] and [3].

There are other estimates of type  $\sup + \inf$  on complex Monge-Ampere equation on compact manifolds, see [21-22]. They consider, on compact Kahler manifold  $(M, g)$ , the following equation:

$$\begin{cases} (\omega_g + \partial\bar{\partial}\varphi)^n = e^{f-t\varphi}\omega_g^n, \\ \omega_g + \partial\bar{\partial}\varphi > 0 \text{ on } M \end{cases}$$

And, they prove some estimates of type  $\sup_M + m \inf_M \leq C$  or  $\sup_M + m \inf_M \geq C$  under the positivity of the first Chern class of  $M$ .

In this paper, we will prove a Harnack type inequality for equation (1). Precisely,

**Theorem 1.1.** *Let  $(M, g)$  be a Riemannian manifold of dimension  $n = 4, 5$  or  $6$  and let  $K \subset M$  be a compact set. Then, there exists a positive constant  $c$ , which depending only on  $h_0 = \|\bar{h}\|_{L^\infty}$ ,  $K$ ,  $M$ , the metric  $g$  and dimension  $n$  such that*

$$\sup_K u \times \inf_M u \leq c, \quad (4)$$

for any solution  $u$  of the equation (1).

Our proof is an extension of Brezis-Li [8] and Li-Zhang result in dimension 3 and 4 [18]. We refer to [18] for exposition on importance of studying these inequalities on manifolds. The proof of Theorem 1.1 relies on moving plane method together with some ideas in [18] and [19]. The equation (1) is first written in local coordinates ([18]) which gives an expression for the Laplace-Beltrami operator as a perturbation of the usual Laplace operator and do blow up analysis. Then constructing suitable auxiliary function, we apply Lemma 2.1 of [19]. Here, we wish to point out that the restriction on dimension  $n$  is due to difficulty in estimating the perturbation terms of the Laplace-Beltrami operator. With finer estimates, the result could possibly be extended to higher dimension.

Note that our result is true for more general equations where  $h$  may not be the scalar curvature. Also, we extend the result of [5], where in the case  $h \equiv \epsilon \in (0, 1)$  and  $u_\epsilon$  solution of

$$-\Delta u_\epsilon + \epsilon u_\epsilon = V_\epsilon u_\epsilon^{N-1}, \quad u_\epsilon > 0. \quad (E_\epsilon)$$

we have

**Corollary 1.2.** *For all compact set  $K$  of  $M$  there is a positive constant  $c$ , which depends only on,  $K, M, g, n$  such that:*

$$\sup_K u_\epsilon \times \inf_M u_\epsilon \leq c,$$

for all  $u$  solution of  $(E_\epsilon)$ .

Now, if we assume  $M$  a compact Riemannian manifold and  $0 < a \leq V_\epsilon \leq b < +\infty$  then

**Theorem 1.3.** (see [3]). *For all positive numbers  $a, b, m$  there is a positive constant  $c$ , which depends only on,  $a, b, m, M, g$  such that:*

$$\epsilon \sup_M u_\epsilon \times \inf_M u_\epsilon \geq c,$$

for all  $u_\epsilon$  solution of  $(E_\epsilon)$  with

$$\max_M u_\epsilon \geq m > 0.$$

As a consequence of the two previous theorems, we can argue as in [3] that

**Theorem 1.4.** *For  $n=4, 5, 6$ , we have*

$$\max_M u_\epsilon \rightarrow 0,$$

and (up to a subsequence),

$$u_\epsilon \equiv \epsilon^{(n-2)/4}.$$

## 2. PROOF OF THE THEOREM 1.1

We claim that for any  $\epsilon > 0$ ,

$$\epsilon^{n-2} \max_{B(0,\epsilon)} u \times \min_{B(0,4\epsilon)} u \leq c = c(a, b, A, M, g). \quad (5)$$

Arguing by contradiction, we assume that there exists a sequence  $\epsilon_k \rightarrow 0$  and solutions  $u_k$  of (1) such that

$$\max_{B(0,\epsilon_k)} u_k \times \min_{B(0,4\epsilon_k)} u_k \geq k \epsilon_k^{2-n}. \quad (6)$$

The proof of the theorem consists of two main steps. The first is blow up analysis where we analyze the consequence of assuming (6) and obtain equations in local coordinates. In the second part, we wish to apply Lemma 2.1 of [20] and the moving plane method. This involves constructing a suitable auxiliary function and obtaining correct estimates.

### Step 1: Blow-up analysis

For some  $\bar{x}_k \in B(0, \epsilon_k)$ ,  $u_k(\bar{x}_k) = \max_{B(0,\epsilon_k)} u_k$ , due to (6),

$$u_k(\bar{x}_k)^2 \epsilon_k \rightarrow +\infty.$$

By a standard selection process, we can find  $x_k \in B(\bar{x}_k, \epsilon_k/2)$  and  $\sigma_k \in (0, \epsilon_k/4)$  satisfying,

$$u_k(x_k)^2 \sigma_k \rightarrow +\infty, \quad (7)$$

$$u_k(x_k) \geq u_k(\bar{x}_k), \quad (8)$$

$$\text{and, } u_k(x) \leq C_1 u_k(x_k), \text{ in } B(x_k, \sigma_k), \quad (9)$$

where  $C_1$  is some universal constant. It follows from above (6), (8) that

$$u_k(x_k) \times \min_{\partial B(x_k, 2\epsilon_k)} u_k \epsilon_k \geq u_k(\bar{x}_k) \times \min_{B(0, 4\epsilon_k)} u_k \epsilon_k \geq k \rightarrow +\infty. \quad (10)$$

Let  $\{z^1, \dots, z^n\}$  denote some geodesic normal coordinates centered at  $x_k$  (we use the exponential map). In the geodesic normal coordinates, the metric  $g$  is given by  $g = g_{ij}(z) dz^i dz^j$  where

$$g_{ij}(z) - \delta_{ij} = O(r^2), \quad g := \det(g_{ij}(z)) = 1 + O(r^2), \quad h(z) = O(1), \quad (11)$$

where  $r = |z|$ . Thus,

$$\Delta_g u = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j u) = \Delta u + b_i \partial_i u + d_{ij} \partial_{ij} u,$$

where

$$b_j = O(r), \quad d_{ij} = O(r^2) \quad (12)$$

We have a new function

$$v_k(y) = M_k^{-1} u_k(M_k^{-2/(n-2)} y) \text{ for } |y| \leq 3\epsilon_k M_k^{2/(n-2)}$$

where  $M_k = u_k(0)$ . From (9) and (10) we have

$$\left. \begin{aligned} \Delta v_k + \bar{b}_i \partial_i v_k + \bar{d}_{ij} \partial_{ij} v_k - \bar{c} v_k + v_k^{N-1} &= 0 \text{ for } |y| \leq 3\epsilon_k M_k^{2/(n-2)} \\ v_k(0) &= 1 \\ v_k(y) &\leq C_1 \text{ for } |y| \leq \sigma_k M_k^{2/(n-2)} \end{aligned} \right\} \quad (13)$$

where  $C_1$  is a universal constant and

$$\bar{b}_i(y) = M_k^{-2/(n-2)} b_i(M_k^{-2/(n-2)} y), \quad \bar{d}_{ij}(y) = d_{ij}(M_k^{-2/(n-2)} y) \quad (14)$$

and,

$$\bar{c}(y) = M_k^{-4/(n-2)} \bar{h}(M_k^{-2/(n-2)} y). \quad (15)$$

We can see that for  $|y| \leq 3\epsilon_k M_k^{2/(n-2)}$ ,

$$|\bar{b}_i(y)| \leq C M_k^{-4/(n-2)} |y|, \quad |\bar{d}_{ij}(y)| \leq C M_k^{-4/(n-2)} |y|^2, \quad |\bar{c}(y)| \leq C M_k^{-4/(n-2)} \quad (16)$$

where  $C$  depends on  $n, M, g$ .

It follows from (13), (14), (15), (16) and the elliptic estimates, that, along a subsequence,  $v_k$  converges in  $C^2$  norm on any compact subset of  $\mathbb{R}^2$  to a positive function  $U$  satisfying

$$\left. \begin{aligned} \Delta U + U^{N-1} &= 0, \text{ in } \mathbb{R}^n, \text{ with } N = \frac{n+2}{n-2} \\ U(0) &= 1, \quad 0 < U \leq 1. \end{aligned} \right\} \quad (17)$$

According to Caffarelli-Gidas-Spruck, see [10] we have an explicit form of  $U$

$$U(y) = (1 + |y|^2)^{-(n-2)/2}. \quad (18)$$

**Step 2: The Kelvin transform and moving-plane method**

For  $x \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}_+^*$ , let,

$$v_k^1(y) := \frac{1}{|y|^{n-2}} v_k \left( e + \frac{y}{|y|^2} \right) \quad (19)$$

denote the Kelvin transformation of  $v_k$  with respect to the unit ball, defined on  $\mathbb{R}^n \cup \{\infty\} \setminus \{0\}$  and  $e = (1, 0, \dots, 0)$ . If we denote

$$\begin{aligned} \bar{U}_0(y) &= \frac{1}{|y|^{n-2}} U \left( e + \frac{y}{|y|^2} \right) \\ &= \left( \frac{1}{(1 + 2y_1 + 2|y|^2)} \right)^{(n-2)/2} \end{aligned} \quad (20)$$

then

$$v_1^k \rightarrow \bar{U}_0 \text{ in } C_{loc}^2(\mathbb{R}^n \cup \{\infty\} \setminus \{0\}). \quad (21)$$

Note that  $\bar{U}_0$  is still a solution of (17) and  $\bar{U}_0(-e/2) = 2^{(n-2)/2}$  is the maximum of  $\bar{U}_0(y)$ . Hence  $v_1^k$  has a non degenerate, local maximum near  $-e/2$  for all large  $k$ . To arrive at a contradiction to our assumption (6), we use the same method as in [19] of moving-plane, precisely Lemma 2.1 in [19] to show that  $\frac{\partial v_k^1}{\partial x_1} < 0$  near the point  $e^* = (-1/2, 0, \dots, 0) = -e/2$ .

Note, that, if we consider  $\tilde{x}_k = \exp_{x_k}(M_k^{-2/(n-2)}e)$ , we work in the conformal coordinates in the exponential map around this point, for the blow-up analysis. A computation gives,

$$\partial_i v_k \left( \frac{y}{|y|^2} \right) = |y|^{n-2} ((n-2)y_i v_k^1(y) + (\delta_{im}|y|^2 - 2y_i y_m) \partial_m v_k^1(y)),$$

and,

$$\partial_{ij} v_k \left( \frac{y}{|y|^2} \right) = \alpha_1 v_k^1(y) + \beta_{1m} \partial_m v_k^1(y) + \gamma_{ml} \partial_{ml} v_k^1(y),$$

with,

$$\alpha_1 = (n-2)(\delta_{ij}|y|^n - ny_i y_j |y|^{n-2}),$$

$$\begin{aligned} \beta_{1m} &= |y|^{n-2} \{ (n-2)(\delta_{im}|y|^2 - 2y_i y_m)(y_i - y_j) \\ &\quad - 2(\delta_{im}y_j |y|^2 + \delta_{ij}y_m |y|^2 + \delta_{jm}y_i |y|^2 - 2y_i y_j y_m) \}, \end{aligned}$$

and,

$$\gamma_{ml} = |y|^{n-2} (\delta_{im}|y|^2 - 2y_i y_m)(\delta_{jl}|y|^2 - 2y_j y_l).$$

The function  $v_k^1$  satisfies the equation:

$$\Delta v_k^1 + (v_k^1)^{N-1} = E_1(y)$$

where,

$$E_1(y) = -\frac{1}{|y|^{n+2}} \left\{ \bar{b}_i \left( \frac{y}{|y|^2} \right) \partial_i v_k \left( e + \frac{y}{|y|^2} \right) + \bar{d}_{ij} \left( \frac{y}{|y|^2} \right) \partial_{ij} v_k \left( e + \frac{y}{|y|^2} \right) - \bar{c} \left( e + \frac{y}{|y|^2} \right) v_k \left( e + \frac{y}{|y|^2} \right) \right\}. \quad (22)$$

Thus,  $v_k^1$  is a solution of an equation

$$\Delta v_k^1 + \tilde{d}_{ml} \partial_{ml} v_k^1 + \tilde{b}_m \partial_m v_k^1 + \tilde{c} v_k^1 + (v_k^1)^{N-1} = 0, \quad (23)$$

with,

$$|\tilde{d}_{ml}| \leq C \frac{M_k^{-4/(n-2)}}{|y|^2}, \quad (24)$$

$$|\tilde{b}_m| \leq C \frac{M_k^{-4/(n-2)}}{|y|^3}, \quad (25)$$

$$\text{and, } |\tilde{c}| \leq C \frac{M_k^{-4/(n-2)}}{|y|^4}. \quad (26)$$

Thus, if we denote

$$L_k := \Delta + \tilde{d}_{ml} \partial_{ml} + \tilde{b}_m \partial_m + \tilde{c}, \quad (27)$$

then for any  $C^2$  function  $g$ ,

$$L_k g \rightarrow \Delta g \text{ as } k \rightarrow \infty \quad (28)$$

if say,  $\partial_{ml} g$ ,  $\partial_m g$  and  $\tilde{c} g$  are all uniformly bounded.

For  $\lambda < 0$ , set  $T_\lambda = \{y, y_1 = \lambda\}$ ,  $\Sigma_\lambda = \{y, y_1 > \lambda\}$  and  $\Sigma'_\lambda = \Sigma_\lambda - \{y : |y| \leq r_k^{-1/2}\}$ . Let

$$v_k^{1,\lambda} = v_k^1(2\lambda - y_1, y_2, \dots, y_n) \quad (29)$$

and,

$$w_\lambda := v_k^1 - v_k^{1,\lambda}.$$

Then,

$$L_k w_\lambda + \tilde{b} w_\lambda = V_\lambda,$$

where,

$$V_\lambda = (\tilde{d}_{ml} - \tilde{d}_{ml}^\lambda) \partial_{ml} v_k^{1,\lambda} + (\tilde{b}_m - \tilde{b}_m^\lambda) \partial_m v_k^{1,\lambda} + (\tilde{c} - \tilde{c}^\lambda) v_k^{1,\lambda}, \quad (30)$$

and  $\tilde{b}(x)$  lies between  $v_k^1(x)$  and  $v_k^{1,\lambda}(x)$ .

From the expression of  $v_k^1$  and  $v_k^{1,\lambda}$ , we have

$$|\partial_m v_k^{1,\lambda}| \leq C \left( \frac{1}{|y^\lambda|^{n-1}} + \frac{1}{|y^\lambda|^n} \right), \quad (31)$$

$$|\partial_{ml} v_k^{1,\lambda}| \leq C \left( \frac{1}{|y^\lambda|^n} + \frac{1}{|y^\lambda|^{n+1}} \right). \quad (32)$$

Hence, for  $y \in \Sigma'_\lambda$ ,

$$\begin{aligned} |\tilde{d}_{ml}\partial_{ml}w_\lambda| &\leq C \frac{M_k^{-4/(n-2)}}{|y|^2} \left( \frac{1}{|y|^n} + \frac{1}{|y|^{n+1}} \right) \\ |\tilde{b}_m\partial_m w_\lambda| &\leq C \frac{M_k^{-4/(n-2)}}{|y|^3} \left( \frac{1}{|y|^{n-1}} + \frac{1}{|y|^n} \right) \\ |\tilde{c}w_\lambda| &\leq C \frac{M_k^{-4/(n-2)}}{|y|^{n-2}}. \end{aligned}$$

Recall that  $\frac{1}{2}R_k M_k^{2/(n-2)} < r_k \leq \eta_0 R_k M_k^{2/(n-2)}$ ,  $R_k \rightarrow 0$  and  $\eta_0$  is a constant chosen later. Observe that for dimensions  $3 \leq n \leq 6$ , all of

$$|\tilde{d}_{ml}\partial_{ml}w_\lambda|, |\tilde{b}_m\partial_m w_\lambda|, |\tilde{c}w_\lambda| \rightarrow 0 \text{ uniformly as } k \rightarrow \infty, \text{ independent of } \lambda < 0. \quad (33)$$

Furthermore,

$$\begin{aligned} |(\tilde{d}_{ml} - \tilde{d}_{ml}^\lambda)\partial_{ml}v_k^{1,\lambda}| &\leq \frac{CM_k^{-4/(n-2)}(y_1 - \lambda)}{|y|^4|y^\lambda|^{n-2}}, \\ |(\tilde{b}_m - \tilde{b}_m^\lambda)\partial_m v_k^{1,\lambda}| &\leq \frac{CM_k^{-4/(n-2)}(y_1 - \lambda)}{|y|^4|y^\lambda|^{n-2}}, \end{aligned}$$

and,

$$|(\tilde{c} - \tilde{c}^\lambda)v_k^{1,\lambda}| \leq \frac{CM_k^{-4/(n-2)}(y_1 - \lambda)}{|y|^4|y^\lambda|^{n-2}}$$

implies that

$$V_\lambda \leq |V_\lambda| \leq \frac{CM_k^{-4/(n-2)}(y_1 - \lambda)}{|y|^4|y^\lambda|^{n-2}}. \quad (34)$$

First of all we have the following lemma as in [19]

**Lemma 2.1.** *There exists constants  $\lambda_0 \leq -2$  and  $c_0 = c_0(n, \mu) > 0$ , both independent of  $k$  such that:*

$$v_k^1(y) - v_k^1(y^{\lambda_0}) \geq c_0(1 + |y|)^{-n}(y_1 - \lambda)$$

for  $y_1 \geq \lambda_0$  and  $|y| \geq r_k^{-1/2}$ .

*Proof.* To prove our estimate, we consider two sets  $|y| \geq \delta > 0$  and  $\delta \geq |y| \geq r_k^{-1/2}$ . For the first set we use the same technique as in Prajapat-Lin paper ([19]), we use the  $C^2$  convergence of  $v_k^1$  to  $\bar{U}_0$  and choose  $|\lambda|$  big enough to have our estimate. For two positive constants  $C_1$  and  $C_2$  we write the estimate as follows:

$$v_k^1(y) - v_k^1(y^\lambda) \geq |\lambda| \frac{C_1}{|y|^n} (y_1 - \lambda) - \frac{C_2}{|y|^n} (y_1 - \lambda).$$

For  $|y| > \delta$ ,  $\left| \frac{y}{|y|^2} \right| = \frac{1}{|y|} < 1/\delta$ . Using convergence of  $v_k$  to  $U$  in  $C^2$  norm in  $B(0, 1/\delta)$  we have  $C' > v_k > C > 0$  for  $|y| > \delta$ . We write

$$\begin{aligned}
& v_k^1(y) - v_k^1(y^\lambda) \\
&= \left( \frac{1}{|y|^{n-2}} - \frac{1}{|y^\lambda|^{n-2}} \right) v_k \left( \frac{y}{|y|^2} \right) \\
&\quad + \frac{1}{|y^\lambda|^{n-2}} \frac{v_k \left( \frac{y}{|y|^2} \right) - v_k \left( \frac{y^\lambda}{|y^\lambda|^2} \right)}{\frac{y}{|y|^2} - \frac{y^\lambda}{|y^\lambda|^2}} \left( \frac{y}{|y|^2} - \frac{y^\lambda}{|y^\lambda|^2} \right) \\
&=: I_1 + I_2.
\end{aligned} \tag{35}$$

Note that

$$\frac{v_k \left( \frac{y}{|y|^2} \right) - v_k \left( \frac{y^\lambda}{|y^\lambda|^2} \right)}{\frac{y}{|y|^2} - \frac{y^\lambda}{|y^\lambda|^2}}$$

appearing in the second term in (35) can be estimated by  $\nabla U$  in the compact set  $|y| \leq \frac{1}{\delta}$  as in [19] again using the  $C^2$  convergence of  $v_k$ . While

$$\frac{y}{|y|^2} - \frac{y^\lambda}{|y^\lambda|^2} = \frac{y - y^\lambda}{|y|^2} + y^\lambda \left( \frac{1}{|y|^2} - \frac{1}{|y^\lambda|^2} \right) = 2 \frac{y_1 - \lambda}{|y|^2} + \frac{4(y_1 - \lambda)y^\lambda}{|y|^2 |y^\lambda|^2}.$$

Since  $y_1 > \lambda$  we have  $|y^\lambda| > -\lambda$  thus,

$$\left| \frac{4(y_1 - \lambda)y^\lambda}{|y|^2 |y^\lambda|^2} \right| < \frac{4(y_1 - \lambda)}{-\lambda |y|^2}.$$

It follows that

$$\frac{y}{|y|^2} - \frac{y^\lambda}{|y^\lambda|^2} \leq \frac{6(y_1 - \lambda)}{|y|^2}$$

and

$$I_2 < \frac{C(y_1 - \lambda)}{|y|^2 |y^\lambda|^{n-2}}. \tag{36}$$

To estimate the first term  $I_1$ , since  $v_k > c > 0$  and  $y_1 > \lambda$ , we have

$$I_1 = \left( \frac{1}{|y|^{n-2}} - \frac{1}{|y^\lambda|^{n-2}} \right) v_k \left( \frac{y}{|y|^2} \right) > c \frac{|y^\lambda|^{n-2} - |y|^{n-2}}{|y|^{n-2} |y^\lambda|^{n-2}}$$

We now use the binomial formula:

$$|y^\lambda|^{n-2} - |y|^{n-2} = (|y^\lambda| - |y|)(|y|^{n-3} + \dots + |y^\lambda|^{n-3}), k = n - 3.$$

Observe that

$$|y^\lambda| - |y| = \frac{|y^\lambda|^2 - |y|^2}{|y| + |y^\lambda|} = \frac{-4\lambda(y_1 - \lambda)}{|y| + |y^\lambda|}.$$



Thus,

$$|y^\lambda|^{n-2} - |y|^{n-2} = \frac{-4\lambda(y_1 - \lambda)(|y|^k + \dots + |y^\lambda|^k)}{|y| + |y^\lambda|}$$

and

$$\left( \frac{1}{|y|^{n-2}} - \frac{1}{|y^\lambda|^{n-2}} \right) v_k \left( \frac{y}{|y|^2} \right) > c \frac{-4\lambda(y_1 - \lambda)(|y|^k + \dots + |y^\lambda|^k)}{(|y| + |y^\lambda|)|y|^{n-2}|y^\lambda|^{n-2}}$$

Finally, because  $|y^\lambda| > |y|$ , we have

$$\frac{|y|^k + \dots + |y^\lambda|^k}{(|y| + |y^\lambda|)|y|^{n-2}|y^\lambda|^{n-2}} > \frac{|y|^{n-4}(|y| + |y^\lambda|)}{(|y| + |y^\lambda|)|y|^{n-2}|y^\lambda|^{n-2}} = \frac{1}{|y|^2|y^\lambda|^{n-2}}.$$

It follows that

$$\left( \frac{1}{|y|^{n-2}} - \frac{1}{|y^\lambda|^{n-2}} \right) v_k \left( \frac{y}{|y|^2} \right) > \frac{c}{|y|^2|y^\lambda|^{n-2}}.$$

Therefore,

$$I_1 + I_2 > c \frac{-4\lambda(y_1 - \lambda)}{|y|^2|y^\lambda|^{n-2}} - C \frac{-4(y_1 - \lambda)}{|y|^2|y^\lambda|^{n-2}}$$

with  $c, C > 0$  and for  $-\lambda$  big enough, we have the required inequality. On the annulus  $A_k(\delta) := \{y : r_k^{-1/2} \leq |y| \leq \delta\}$ , using the maximum principle for  $v_k^1$  we have

$$\min_{A_k(\delta)} v_k^1 = \min_{\partial A_k(\delta)} v_k^1 \quad (37)$$

where the boundary  $\partial A_k(\delta)$  is the union of two set  $|y| = r_k^{-1/2}$  and  $|y| = \delta$ . From (20),  $\bar{U}_0(0) = 1$ . Hence for given  $\varepsilon > 0$  small, there exists  $\delta_0 > 0$  such that for all  $|y| < \delta_0$ ,

$$1 - \varepsilon = \bar{U}_0(0) - \varepsilon < \bar{U}_0(y) < 1 + \varepsilon = \bar{U}_0(0) + \varepsilon. \quad (38)$$

Choosing  $\varepsilon$  sufficiently small, we have

$$\bar{U}_0(y) > (1 - \frac{\varepsilon}{2}) \text{ for all } |y| \leq \delta_0. \quad (39)$$

While, for  $y \in B(0, \delta)$ ,  $y^\lambda \in B(0_\lambda, \delta)$  where  $0_\lambda := (2\lambda, 0, \dots, 0)$  is the reflection of origin. We have

$$\begin{aligned} \bar{U}_0(y^\lambda) &= \frac{1}{(1 + 2(2\lambda - y_1) + 2|y^\lambda|^2)^{(n-2)/2}} \\ &\leq \frac{1}{(1 + 2(2\lambda - y_1) + (2\lambda - y_1)^2)^{(n-2)/2}} \\ &\leq \frac{1}{(1 + 2(2\lambda - \delta) + (2\lambda - \delta)^2)^{(n-2)/2}} \\ &= \frac{1}{(1 - 2(\delta - 2\lambda) + (\delta - 2\lambda)^2)^{(n-2)/2}} \\ &= \frac{1}{(\delta - 2\lambda - 1)^{(n-2)}} \end{aligned} \quad (40)$$

for  $\lambda \leq -2$  and  $0 < \delta \leq \delta_0$ . From  $C^2$  convergence of  $v_k^1$  to  $\bar{U}_0$  in  $B(0_\lambda, \delta)$ , we have

$$v_k^1(y^\lambda) < (1 + \frac{\varepsilon}{2})\bar{U}_0(y^\lambda) \text{ in } B(0, \delta). \quad (41)$$

Note that

$$\begin{aligned} \min_{\{|y|=r_k^{-1/2}\}} v_k^1 &= r_k^{(n-2)/2} \min_{\{|y|=r_k^{1/2}\}} v_k \\ &\geq (1 + \epsilon) r_k^{(n-2)/2} \min_{\{|y|=r_k^{1/2}\}} U(e + y) \\ &= (1 + \epsilon) \min_{\{|y|=r_k^{-1/2}\}} \bar{U}_0(y) \\ &\geq (1 + \frac{\epsilon}{2})\bar{U}_0(0) = (1 + \frac{\epsilon}{2}). \end{aligned} \quad (42)$$

Using  $C^2$  convergence of  $v_k^1$  to  $\bar{U}_0$  on the compact set  $|y| = \delta$  we have,

$$\min_{\{|y|=\delta\}} v_k^1 \geq (1 - \frac{\epsilon}{10}) \min_{\{|y|=\delta\}} \bar{U}_0. \quad (43)$$

In either case, for  $\delta \leq \delta_0$ ,

$$\begin{aligned} \min_{A_k(\delta)} v_k^1 &\geq (1 + \frac{\epsilon}{2})\bar{U}_0(0) = (1 + \frac{\epsilon}{2}) \\ &\geq (1 + \frac{\epsilon}{2})\bar{U}_0(y^\lambda) \\ &> (1 + \frac{\epsilon}{4})v_k^1(y^\lambda) + \varepsilon/10 \end{aligned} \quad (44)$$

□

Recall that  $w_\lambda$  satisfies

$$L_k w_\lambda + \tilde{b} w_\lambda = V_\lambda \text{ in } \Sigma'_\lambda \quad (45)$$

where

$$V_\lambda \leq \frac{C M_k^{-4/(n-2)} (y_1 - \lambda)}{|y|^4 |y^\lambda|^{n-2}}. \quad (46)$$

Now, consider the "auxiliary function"

$$h_\lambda = A r_k^{(2-n)/2} G^\lambda(y, 0) - \int_{\Sigma_\lambda} G^\lambda(y, \eta) \tilde{Q}_\lambda(\eta) d\eta, \quad (47)$$

with

$$\tilde{Q}_\lambda := \frac{C_1 M_k^{-4/(n-2)}}{(|y| + r_k^{-1/2})^4 (|y| - \lambda)^{(n-2)}} = C_1 M_k^{-4/(n-2)} Q_\lambda \quad (48)$$

where we define

$$Q_\lambda = \frac{1}{(|y| + r_k^{-1/2})^4 (|y| - \lambda)^{(n-2)}}$$

for simplicity of notations. Here we choose constant  $C_1 > 0$  (big enough) later. Note that

$$\Delta h_\lambda = \tilde{Q}_\lambda = \frac{C_1 M_k^{-4/(n-2)}}{(|y| + r_k^{-1/2})^4 (|y| - \lambda)^{(n-2)}}.$$

It can be verified that

$$V_\lambda \leq \frac{C M_k^{-4/(n-2)} (y_1 - \lambda)}{|y|^4 |y^\lambda|^{n-2}} \leq \tilde{Q}_\lambda. \quad (49)$$

The function  $h_k^\lambda$  satisfies the following properties:

**Lemma 2.2.** *The functions  $\tilde{b}$  and  $h_\lambda$  satisfy the following properties*  
(i) *For all  $\lambda_0 \leq \lambda \leq -\frac{1}{4}$ ,*

$$0 \leq \tilde{b}(y) \leq \frac{C}{|y|^4} \text{ in } \Sigma'_\lambda \quad (50)$$

(ii) *The auxiliary function  $h_k^\lambda(x) = 0$  on  $x_1 = \lambda$ ,  $\lambda_0 \leq \lambda \leq -\frac{1}{4}$  and  $h_k^\lambda(x) = O(|x|^{-\tau})$  for a constant  $\tau > 0$ ;*  
(iii)  *$h_k^\lambda(x) \in C^1(\Sigma'_\lambda)$  and*

$$0 < h_\lambda(y) \leq A r_k^{-(n-2)/2} G^\lambda(y, 0) + C'_1 M_k^{-2/(n-2)} \frac{(y_1 - \lambda)}{|y|^n} o(1) \quad (51)$$

$$L_k h_\lambda \geq C_1 \tilde{Q}_\lambda. \quad (52)$$

(iv)  *$h_k^{\lambda_0}(x) \leq w_{\lambda_0}$  and that both  $h_k^\lambda$  and  $\nabla_x h_k^\lambda$  are continuous with respect to both the variables  $x$  and  $\lambda$  in  $\Sigma'_\lambda$ . Moreover,*

$$L_k w_\lambda + \tilde{b} w_\lambda \leq C_1 \tilde{Q}_\lambda \leq L_k h_\lambda \quad (53)$$

and hence,

$$L_k(w_\lambda - h_\lambda) + \tilde{b}(w_\lambda - h_\lambda) \leq -\tilde{b} h_\lambda \leq 0. \quad (54)$$

Most of the estimates mentioned from (i)-(iv) above are similar, but much simpler than those in section 4 of [19], and we refer the reader to that paper for details.

However, observe that we cannot apply Lemma 4.1 of [19] directly, as our operator is  $L_k$ . Here the crucial step is to have correct estimates for the perturbation terms in  $L_k h_k^\lambda$ , which we obtain in the following lemmas.

### 3. ESTIMATES FOR $L_k h_k^\lambda$

**Lemma 3.1.** ( **Estimate of  $G^\lambda(y, 0)$**  ) *The function  $G^\lambda(y, 0)$  satisfies*

$$|\tilde{d}_{ml} \partial_{ml} G^\lambda(y, 0) + \tilde{b}_m \partial_m G^\lambda(y, 0) + \tilde{c} G^\lambda(y, 0)| \leq r_k^{(n-2)/2} \frac{C M_k^{-4/(n-2)}}{|y|^4 |y^\lambda|^{n-2}}, \quad (55)$$

and hence

$$L_k G^\lambda(y, 0) \leq C_2 r_k^{(n-2)/2} Q_\lambda. \quad (56)$$

*Proof.* From the fact that,

$$G^\lambda(y, 0) = c_n(|y|^{2-n} - |y^\lambda|^{2-n}),$$

we have, around 0 and  $+\infty$ ,

$$|\tilde{c}G^\lambda(y, 0)| \leq \frac{CM_k^{-4/(n-2)}}{|y|^4|y|^{n-2}} + \frac{CM_k^{-4/(n-2)}}{|y|^4|y^\lambda|^{n-2}},$$

$$|\tilde{b}_m \partial_m G^\lambda(y, 0)| \leq \frac{CM_k^{-4/(n-2)}}{|y|^3|y|^{n-1}} + \frac{CM_k^{-4/(n-2)}}{|y|^3|y^\lambda|^{n-1}},$$

and,

$$|\tilde{d}_{ml} \partial_{ml} G^\lambda(y, 0)| \leq \frac{CM_k^{-4/(n-2)}}{|y|^2|y|^n} + \frac{CM_k^{-4/(n-2)}}{|y|^2|y^\lambda|^n},$$

For the previous expression, we remark that, around 0

$$|y|^3|y|^{n-1} = |y|^2|y|^n = |y|^4|y|^{n-2} \geq |y|^4 r_k^{-(n-2)/2},$$

Thus,

$$|\tilde{d}_{ml} \partial_{ml} G^\lambda(y, 0) + \tilde{b}_m \partial_m G^\lambda(y, 0) + \tilde{c}G^\lambda(y, 0)| \leq r_k^{(n-2)/2} \frac{CM_k^{-4/(n-2)}}{|y|^4|y^\lambda|^{n-2}},$$

and hence

$$L_k G^\lambda(y, 0) \leq C_2 r_k^{(n-2)/2} \tilde{Q}_\lambda$$

□

Now, we look at the second term in  $h_\lambda$  by setting,

$$u = - \int_{\Sigma'_\lambda} G^\lambda(y, \eta) Q_\lambda(\eta) d\eta$$

We have that  $u$  is a solution of a Dirichlet problem on  $\Sigma'_\lambda$ .

**Lemma 3.2. ( Estimate for  $u$  )** *For the function  $u$ , we have*

$$\Delta u = C_1 Q_\lambda \tag{57}$$

$$|\tilde{d}_{ml} \partial_{ml} u + \tilde{b}_m \partial_m u + \tilde{c}u| = o(1) Q_\lambda. \tag{58}$$

*Proof.* We want to prove that,

$$|\tilde{d}_{ml} \partial_{ml} u + \tilde{b}_m \partial_m u + \tilde{c}u| = o(1) Q_\lambda$$

Because of the expression of  $G^\lambda$ , we consider  $u$  as a difference of two convolution product on  $\Sigma_\lambda$ . Thus, to differentiate  $u$  is equivalent to differentiating  $Q_\lambda$  inside the integral plus the boundary term. Our aim is to estimate the auxiliary function and its derivatives of order less than two near infinity. Write,

$$-\tilde{u} = \int_{\Sigma_\lambda} G^\lambda(y, \eta) Q_\lambda(\eta) d\eta$$

and,

$$u - \tilde{u} = - \int_{B(0, r_k^{-1/2})} G^\lambda(y, \eta) Q_\lambda(\eta) d\eta$$

We denote  $\Sigma_\lambda^s$  the reflection of  $\Sigma_\lambda$ , with respect to the hyperplane  $T_\lambda = \{y_1 = \lambda\}$ , i.e.,  $\Sigma_\lambda^s := \{y \in \mathbb{R}^n : y_1 < \lambda\}$ . Thus,

$$\begin{aligned} -\tilde{u} &= \int_{\Sigma_\lambda} c_n(|y - \eta|^{2-n} - |y - \eta^\lambda|^{2-n}) Q_\lambda(\eta) d\eta \\ &= \int_{\Sigma_\lambda} |y - \eta|^{2-n} Q_\lambda(\eta) d\eta - \int_{\Sigma_\lambda^s} |y - \eta|^{2-n} Q_\lambda(\eta^\lambda) d\eta, \\ &= \int_{\mathbb{R}^n} |y - \eta|^{2-n} Q_\lambda(\eta) d\eta - \int_{\Sigma_\lambda^s} |y - \eta|^{2-n} (Q_\lambda(\eta) + Q_\lambda(\eta^\lambda)) d\eta \\ &=: f + v \end{aligned} \tag{60}$$

where

$$f \in C^\infty(\mathbb{R}^n), \quad \Delta f = Q_\lambda \text{ in } \mathbb{R}^n \tag{61}$$

$$\text{and } \Delta v = 0 \text{ in } \Sigma_\lambda. \tag{62}$$

**Behavior of  $\tilde{u}$  and  $u - \tilde{u}$  near infinity:** Let  $\varphi$  a cutoff function in the unit ball, i.e.,

$$0 \leq \varphi \leq 1, \quad \varphi \equiv 1 \text{ on } B(0, 1/2) \text{ and } \varphi \equiv 0 \text{ in } \mathbb{R}^n \setminus B(0, 1)$$

We write,

$$f = f_1 + f_2,$$

where,

$$\begin{aligned} f_1 &= \int_{\mathbb{R}^n} |y - \eta|^{2-n} Q_\lambda(\eta) \varphi d\eta \text{ and} \\ f_2 &= \int_{\mathbb{R}^n} |y - \eta|^{2-n} Q_\lambda(\eta) (1 - \varphi) d\eta. \end{aligned}$$

Then,

$$\Delta f_1 = Q_\lambda \varphi \text{ and } \Delta f_2 = Q_\lambda (1 - \varphi),$$

We use the Fourier transform to prove that  $f_2$  is in the Schwartz space and thus,

$$|f_2| \leq C|y|^{2-n}, \quad |\partial f_2| \leq C|y|^{1-n}, \quad |\partial^2 f_2| \leq C|y|^{-n}$$

For  $f_1$  we use the fact that in the present case,  $|y|$  is big enough, and we differentiate inside the integral, ( $|\eta| < |y|/2 \Rightarrow |y - \eta| > |y|/2$ ) to conclude

$$|f_1| \leq C|y|^{2-n}, \quad |\partial f_1| \leq C|y|^{1-n}, \quad |\partial^2 f_1| \leq C|y|^{-n}$$

Thus, for  $|y|$  large,

$$|f| \leq C|y|^{2-n}, \quad |\partial f| \leq C|y|^{1-n}, \quad |\partial^2 f| \leq C|y|^{-n}. \tag{63}$$

For  $v$ , we have  $y \in \Sigma_\lambda$  and the integral is taken over the reflected set  $\Sigma_\lambda^s$ . We set,

$$R_\lambda(\eta) = Q_\lambda(\eta) + Q_\lambda(\eta^\lambda),$$

so that

$$\begin{aligned} v(y) &= \int_{\Sigma_\lambda^s} |y - \eta|^{2-n} R_\lambda(\eta) d\eta, \\ &= \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} |y - \eta|^{2-n} R_\lambda(\eta) d\eta - \int_{\Sigma_\lambda^s \cap \{|\eta| \leq |y|/2\}} |y - \eta|^{2-n} R_\lambda(\eta) d\eta. \end{aligned} \quad (64)$$

**Second integral:** If  $|\eta| \leq |y|/2$  then  $|y - \eta| \geq |y|/2$  and thus,

$$\left| \int_{\Sigma_\lambda^s \cap \{|\eta| \leq |y|/2\}} |y - \eta|^{2-n} R_\lambda(\eta) d\eta \right| \leq \begin{cases} Cr_k |y|^{2-n} & \text{if } n = 4 \\ C|y|^{2-n} & \text{if } n = 5, 6. \end{cases} \quad (65)$$

**First integral:** First, we have,

$$|R_\lambda(\eta)| \leq C(1 + |\eta|)^{-2-n},$$

If  $|y - \eta| \leq |y|/2$ , then,

$$\begin{aligned} &\left| \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} |y - \eta|^{2-n} R_\lambda(\eta) d\eta \right| \\ &\leq |R_\lambda(|y|/2)| \int_{\{|\eta - y| \leq |y|/2\}} |y - \eta|^{2-n} d\eta \leq C(1 + |y|)^{-n}, \end{aligned}$$

if  $|y|/2 \leq |y - \eta| \leq 3|y|$ ,

$$\begin{aligned} &\left| \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} |y - \eta|^{2-n} R_\lambda(\eta) d\eta \right| \\ &\leq |R_\lambda(|y|/2)| \int_{\{|y|/2 \leq |\eta - y| \leq 3|y|\}} |y - \eta|^{2-n} d\eta \\ &\leq C(1 + |y|)^{-n}, \end{aligned}$$

and for  $3|y| \leq |y - \eta|$ ,  $|\eta| = |y - \eta - y| \geq 2|y|$ , and thus,  $|\eta - y| = |\eta|(|\theta_\eta - \frac{y}{|\eta|}| \geq |\eta|/2$ . With  $|\theta_\eta| = 1$  the angular part of  $\eta$ . Thus

$$\begin{aligned}
& \left| \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} |y - \eta|^{2-n} R_\lambda(\eta) d\eta \right| \\
& \leq C|y|^{1-n} \int_{\{|\eta| \geq |y|/2\}} |\eta|^{-1-n} d\eta \\
& = C|y|^{1-n} \int_{\{r \geq |y|/2\}} r^{-2} dr \\
& = C|y|^{-n}.
\end{aligned}$$

Thus, in this case too, we have

$$|\partial_i v| \leq \begin{cases} Cr_k |y|^{2-n} & \text{if } n = 4 \\ C|y|^{2-n} & \text{if } n = 5, 6. \end{cases} \quad (66)$$

**Estimate of the first derivatives of  $v$ :** We have,

$$\partial_i v = \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} (y_i - \eta_i) |y - \eta|^{-n} R_\lambda(\eta) d\eta - \int_{\Sigma_\lambda^s \cap \{|\eta| \leq |y|/2\}} (y_i - \eta_i) |y - \eta|^{-n} R_\lambda(\eta)$$

**Second integral:** If  $|\eta| \leq |y|/2$  then  $|y - \eta| \geq |y|/2$  and thus,

$$\left| \int_{\Sigma_\lambda^s \cap \{|\eta| \leq |y|/2\}} (y_i - \eta_i) |y - \eta|^{-n} R_\lambda(\eta) d\eta \right| \leq \begin{cases} Cr_k |y|^{1-n} & \text{if } n = 4, \\ C|y|^{1-n} & \text{if } n = 5, 6. \end{cases} \quad (67)$$

**First integral:** First, we have,

$$|R_\lambda(\eta)| \leq C(1 + |\eta|)^{-2-n},$$

If  $|y - \eta| \leq |y|/2$ , then,

$$\begin{aligned}
& \left| \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} (y_i - \eta_i) |y - \eta|^{-n} R_\lambda(\eta) d\eta \right| \\
& \leq |R_\lambda(|y|/2)| \int_{\{|\eta - y| \leq |y|/2\}} |y - \eta|^{1-n} d\eta \leq C(1 + |y|)^{-1-n}.
\end{aligned}$$

If  $|y|/2 \leq |y - \eta| \leq 3|y|$ ,

$$\begin{aligned} & \left| \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} (y_i - \eta_i) |y - \eta|^{-n} R_\lambda(\eta) d\eta \right| \\ & \leq |R_\lambda(|y|/2)| \int_{\{|y|/2 \leq |\eta - y| \leq 3|y|\}} |y - \eta|^{1-n} d\eta \leq C(1 + |y|)^{-1-n}. \end{aligned}$$

For  $3|y| \leq |y - \eta|$ ,  $|\eta| = |y - \eta - y| \geq 2|y|$ , and thus,  $|\eta - y| = |\eta|(|\theta_\eta - y/|\eta|| \geq |\eta|/2$ . With  $|\theta_\eta| = 1$  the angular part of  $\eta$ . Thus

$$\begin{aligned} \left| \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} (y_i - \eta_i) |y - \eta|^{-n} R_\lambda(\eta) d\eta \right| & \leq C|y|^{-n} \int_{\{|\eta| \geq |y|/2\}} |\eta|^{-1-n} d\eta \\ & = C|y|^{-n} \int_{\{r \geq |y|/2\}} r^{-2} dr \\ & = C|y|^{-1-n}. \end{aligned}$$

Thus,

$$|\partial_i v| \leq \begin{cases} Cr_k |y|^{1-n} & \text{if } n = 4 \\ C|y|^{1-n} & \text{if } n = 5, 6. \end{cases}$$

**Estimates for the second derivatives:** We write,

$$\begin{aligned} \partial_{ij} v &= \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} \partial_j((y_i - \eta_i) |y - \eta|^{-n}) R_\lambda(\eta) d\eta \\ &\quad - \int_{\Sigma_\lambda^s \cap \{|\eta| \leq |y|/2\}} \partial_j((y_i - \eta_i) |y - \eta|^{-n}) R_\lambda(\eta). \end{aligned}$$

**Second integral:** If  $|\eta| \leq |y|/2$  then  $|y - \eta| \geq |y|/2$  and  $|\partial_j((y_i - \eta_i) |y - \eta|^{-n})| \leq |y - \eta|^{-n}$ , thus,

$$\left| \int_{\Sigma_\lambda^s \cap \{|\eta| \leq |y|/2\}} \partial_j((y_i - \eta_i) |y - \eta|^{-n}) R_\lambda(\eta) d\eta \right| \leq \begin{cases} Cr_k |y|^{-n} & \text{if } n = 4 \\ C|y|^{-n} & \text{if } n = 5, 6. \end{cases}$$

**First integral:** We use an integration by part,

$$\begin{aligned} & \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} \partial_j \frac{(y_i - \eta_i)}{|y - \eta|^{-n}} R_\lambda(\eta) d\eta = \\ & - \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} \frac{(y_i - \eta_i)}{|y - \eta|^{-n}} \partial_j R_\lambda(\eta) d\eta + \int_{\partial(\Sigma_\lambda^s \cap \{|\sigma| \geq |y|/2\})} \frac{(y_i - \sigma_i)}{|y - \sigma|^{-n}} R_\lambda(\sigma) \nu_j(\sigma) d\sigma. \end{aligned}$$



From, the computation for the first derivatives, we have,

$$\left| \int_{\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}} \frac{(y_i - \eta_i)}{|y - \eta|^{-n}} \partial_j R_\lambda(\eta) d\eta \right| \leq C|y|^{-n},$$

The boundary term has the following decomposition,  $\partial(\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}) = (\partial\Sigma_\lambda^s \cap \{|\eta| \geq |y|/2\}) \cup (\Sigma_\lambda^s \cap \{|\eta| = |y|/2\})$ . For the first boundary,  $\nu_j(\sigma) = 0$  for  $j \neq 1$ , and thus,

$$\int_{\partial(\Sigma_\lambda^s \cap \{|\sigma| \geq |y|/2\})} \frac{(y_i - \sigma_i)}{|y - \sigma|^{-n}} R_\sigma(\sigma) \nu_j(\sigma) d\sigma = \int_{\Sigma_\lambda^s \cap \{|\sigma| = |y|/2\}} \frac{(y_i - \sigma_i)}{|y - \sigma|^{-n}} R_\sigma(\sigma) \nu_j(\sigma) d\sigma$$

Clearly, we have,

$$\int_{\Sigma_\lambda^s \cap \{|\sigma| = |y|/2\}} \frac{|y_i - \sigma_i|}{|y - \sigma|^{-n}} |R_\sigma(\sigma)| |\nu_j(\sigma)| d\sigma \leq C|y|^{-n},$$

Thus, for  $j \neq 1$ , we have:

$$|\partial_{ij} v| \leq \begin{cases} Cr_k |y|^{-n} & \text{if } n = 4 \\ C|y|^{-n} & \text{if } n = 5, 6. \end{cases}$$

But,  $\Delta v = 0$ , thus,

$$|\partial_{11} v| = \left| \sum_{i=2}^n \partial_{ii} v \right| \leq \begin{cases} Cr_k |y|^{-n} & \text{if } n = 4 \\ C|y|^{-n} & \text{if } n = 5, 6. \end{cases}$$

Finally, we have:

$$|\partial_i u| \leq \begin{cases} Cr_k |y|^{1-n} & \text{if } n = 4 \\ C|y|^{1-n} & \text{if } n = 5, 6. \end{cases}$$

and,

$$|\partial_{ij} u| \leq \begin{cases} Cr_k |y|^{-n} & \text{if } n = 4, \\ C|y|^{-n} & \text{if } n = 5, 6. \end{cases}$$

**Estimate for  $u - \tilde{u}$ :** Around infinity, we use the fact that,  $y$  is big enough ( $|\eta| < |y|/2 \Rightarrow |y - \eta| > |y|/2$ ) and we differentiate inside the integral, to have:

$$|u - \tilde{u}| \leq C|y|^{2-n}, \quad |\partial(u - \tilde{u})| \leq C|y|^{1-n}, \quad |\partial^2(u - \tilde{u})| \leq C|y|^{-n}. \quad (68)$$

**Behavior of  $u$  and  $u - \tilde{u}$  near 0:** The function  $f_2$  is smooth and solution of an elliptic equation with  $Q_\lambda(1 - \varphi) \in C^\infty(\mathbb{R}^n)$ , thus, by the elliptic estimates, we have,

$$\|f_2\|_{C^2(B(0,1))} \leq C,$$

We write the function  $f_1$  as,

$$f_1(y) = \int_{B(0,1)} |y - \eta|^{2-n} Q_\lambda(\eta) \varphi d\eta$$

note that,  $|Q_\lambda(\eta)| \leq Cr_k^2$  and thus,

$$|f_1(y)| \leq Cr_k^2$$

Moreover, we can write

$$\partial_i f_1 = \int_{B(0,1)} (y_i - \eta_i) |y - \eta|^{-n} Q_\lambda(\eta) \varphi d\eta$$

Thus,

$$|\partial_i f_1| \leq \int_{B(0,1)} |y - \eta|^{1-n} Q_\lambda(\eta) \varphi d\eta \leq Cr_k^2,$$

Also, we can write, (see, Gilbarg-Trudinger),

$$\begin{aligned} \partial_{ij} f_1(y) &= \int_{B(0,1)} \partial_j ((y_i - \eta_i) |y - \eta|^{-n}) (Q_\lambda(\eta) \varphi(\eta)) d\eta \\ &\quad + Q_\lambda(y) \varphi(y) \int_{\partial B(0,1)} (y_i - \sigma_i) |y - \sigma|^{-n} d\sigma. \end{aligned}$$

Thus,

$$|\partial_{ij} f_1(y)| \leq C \int_{B(0,1)} |y - \eta|^{-n} |Q_\lambda(\eta) \varphi(\eta) - Q_\lambda(y) \varphi(y)| d\eta + Cr_k^2$$

We write,

$$\begin{aligned} &\int_{B(0,1)} \frac{|Q_\lambda(\eta) \varphi(\eta) - Q_\lambda(y) \varphi(y)|}{|y - \eta|^n} d\eta = \\ &\int_{B(0,1) \cap \{|\eta| \geq |y|/2\}} \frac{|Q_\lambda(\eta) \varphi(\eta) - Q_\lambda(y) \varphi(y)|}{|y - \eta|^n} d\eta \\ &+ \int_{B(0,1) \cap \{|\eta| \leq |y|/2\}} \frac{|Q_\lambda(\eta) \varphi(\eta) - Q_\lambda(y) \varphi(y)|}{|y - \eta|^n} d\eta. \end{aligned}$$

**Second integral:** We have  $|\eta| \leq |y|/2$ , thus  $|y - \eta| \geq |y|/2$ , and thus,

$$\begin{aligned} &\int_{B(0,1) \cap \{|\eta| \leq |y|/2\}} |y - \eta|^{-n} |Q_\lambda(\eta) \varphi(\eta) - Q_\lambda(y) \varphi(y)| \\ &\leq \frac{Cr_k^2}{|y|} \int_{B(0,1) \cap \{|\eta| \leq |y|/2\}} |y - \eta|^{1-n} d\eta \\ &\leq \frac{Cr_k^2}{|y|}. \end{aligned}$$

**First integral:** We write,

$$Q_\lambda(\eta) \varphi(\eta) - Q_\lambda(y) \varphi(y) = (\eta - y) \nabla Q_\lambda(\xi), \quad \text{with, } \xi \text{ between } \eta \text{ and } y,$$

We remark that,

$$|\nabla Q_\lambda(\xi)| \leq C(|\xi| + r_k^{-1/2})^{-5},$$

If  $|y| \leq |\xi| \leq |\eta|$ , then,

$$|\nabla Q_\lambda(\xi)| \leq C(|y| + r_k^{-1/2})^{-5} \leq Cr_k^2/|y|$$

If  $|y|/2 \leq |\eta| \leq |\xi| \leq |y|$ ,

$$|\nabla Q_\lambda(\xi)| \leq C(|\eta| + r_k^{-1/2})^{-5} \leq Cr_k^2/|y|$$

Finally, we have:

$$|\partial_{ij} f_1(y)| \leq Cr_k^2/|y|$$

Now, we estimate  $v$  near 0, as for  $f$  we decompose  $v$  in two functions  $v_1$  and  $v_2$ , and we see that  $y \in \Sigma_\lambda$  small enough is far from the symmetral  $\Sigma_\lambda^s$  of  $\Sigma_\lambda$ . And we differentiate inside the integral to have:

$$|\partial_{ij} v_1(y)| \leq Cr_k^2 \text{ and } |\partial_{ij} v_2(y)| \leq C.$$

Now, for  $u - \tilde{u}$ , we use the fact that  $|y| \geq \sigma r_k^{-1/2}$  with  $\sigma > 1$  and the elliptic interior estimates to have (we differentiate inside the integral)

$$|u - \tilde{u}|_{C^0(B(0,1))} \leq Cr_k, \quad |u - \tilde{u}|_{C^1(B(0,1))} \leq Cr_k^{3/2}, \quad |u - \tilde{u}|_{C^2(B(0,1))} \leq Cr_k^2$$

It follows that

$$|\tilde{d}_{ml}\partial_{ml}u + \tilde{b}_m\partial_mu + \tilde{c}u| = o(1)Q_\lambda \quad (69)$$

and that

$$L_k u = (C_1 + o(1))Q_\lambda. \quad (70)$$

□

As in [19], we have the following lemma (which we state without proof)

**Lemma 3.3.** . For  $Q_\lambda$ , we have in  $\Sigma'_\lambda$  for  $\lambda \leq -1/4$  and for large  $k$ :

$$r_k^{(n-2)/2} \int_{\Sigma'_\lambda} G^\lambda(y, \eta) \tilde{Q}_\lambda(\eta) d\eta = o(1)G^\lambda(y, 0).$$

If , we choose  $A > 0$  small enough in the definition of  $h_\lambda$ , we have,

$$h_\lambda > 0,$$

$$L_k h_\lambda \geq C_1 \tilde{Q}_\lambda.$$

We can now use Lemma 4.1 of [19] to obtain a contradiction, and this completes the proof of Theorem 1.1.

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HARNACK TYPE INEQUALITY ON RIEMANNIAN MANIFOLDS OF DIMENSIONS 4, 5 AND  $2n$

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